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NEIGHBORLY COMBINATORIAL 3-MANIFOLDS WITH DIHEDRAL AUTOMORPHISM GROUP

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ABSTRACT

It is well known that for any $n \geqq 5$ the boundary complex of the cyclic 4-polytope $C(n,4)$ is a neighborly combinatorial 3-sphere admitting a vertex transitive action of the dihedral group D_n of order $2n$. In this paper we present a similar series of neighborly combinatorial 3-manifolds with $n \geqq 9$ vertices, each homeomorphic to the "3-dimensional Klein bottle". For $n = 9$ and $n = 10$ these examples have been observed. before by A. Altshuler and L. Steinberg. Moreover we give a computer-aided enumeration of all neighborly combinatorial 3-manifolds with such a symmetry and with at most 19 vertices. It turns out that there are only four other types, one with 10, 15, 17, 19 vertices. We also discuss the more general case of manifolds with cyclic automorphism group C_n.

1. Introduction and results

A *combinatorial* 3-*manifold* M is a simplical complex such that for each vertex $v \in M$ the link of v is a triangulated 2-sphere (cf. [16] ch. 23). It is called *neighborly* if it contains all possible edges, i.e. if for any pair v_1, v_2 of vertices the edge joining them belongs to the simplicial complex. This means that each vertex link contains all the other vertices. Neighborly combinatorial manifolds are of particular interest because of their relationship with the lower and the upper bound conjecture for combinatorial manifolds (cf. [17], [20]). The most classical examples are the boundary complexes of the *cyclic polytopes* discovered and studied by several authors (see [8]). In particular the boundary complex of any cyclic 4-polytope $C(n,4)$ spanned by the n vertices

$$v_i = \left(\cos\frac{2i\pi}{n}, \sin\frac{2i\pi}{n}, \cos\frac{4i\pi}{n}, \sin\frac{4i\pi}{n}\right), \qquad i = 1,\dots,n, \quad n \geqq 5$$

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is a *neighborly combinatorial* 3-*sphere* S^3. According to *Gale's evenness condition* (see [8]) it contains exactly all the 3-dimensional faces $v_iv_{i+1}v_jv_{j+1}$ where the addition of indices is taken modulo n. In the following we denote the vertices simply by i instead of v_i. Consequently the cyclic group C_n of order n acts on the cyclic polytope (transitively on the set of vertices). In cycle notation we denote the generator of this group by $T = (1\ 2\ 3 \dots n)$. Moreover on the cyclic polytope there acts the reflection $R = (1\ \ n-1)\ (2\ \ n-2)\ (3\ \ n-3) \dots$. Therefore the boundary complex of any cyclic 4-polytope is invariant under the action of the dihedral group D_n of order $2n$ generated by T and R. (In fact this is true for cyclic polytopes of arbitrary even dimension; however, throughout this paper we want to restrict ourselves to the case of 3-manifolds.) For the related notion of a block design the invariance under T is called "cyclic design", the invariance under T and R is called "symmetric cyclic design" (cf. [10]).

THEOREM 1 (well known). *For any $n \geqq 5$ there exists a neighborly combinatorial 3-sphere with n vertices which is invariant under the vertex transitive action of the dihedral group D_n: the boundary complex of the cyclic 4-polytope with n vertices. Notation: type I_n.*

For up to 10 vertices all combinatorial types of neighborly 3-manifolds have been enumerated by several authors. According to [2], [3], [4], [6] these are the following:

$n \leqq 7$: 1 type,

$n = 8$: 4 types (1 with dihedral group D_8),

$n = 9$: 51 types (2 with dihedral group D_9),

$n = 10$: 3677 types (5 with cyclic group C_{10}, 3 with dihedral group D_{10}).

The dihedral automorphism group has the largest order among all groups which occur for those types (however, for $n = 10$ there occurs a type N^{10}_{425} invariant under the action of a nondihedral metacyclic group of order 20, see [2], remark 10). On the other hand an enumeration of all types with $n = 11$ vertices seems to be hopeless (cf. [2] Remark 5). For that reason we tried to give an enumeration of all neighborly combinatorial 3-manifolds with dihedral automorphism group D_n. This work has been done by a computer program and carried out for $n \leqq 19$ (for details see below). A similar program gave an enumeration for $n \leqq 15$ under the weaker assumption of a cyclic automorphism group C_n (see section 5 below). A particular topological type came out for any

$n \leqq 19$. It is the "3-dimensional Klein bottle" K^3 defined by $K^3 := S^2 \times [0,1]/\sim$ with $(x,0) \sim (rx,1)$ where $r: S^2 \to S^2$ is the reflection at the equatorial plane of the 2-sphere S^2. It may be described also as the total space of the only nontrivial S^2-bundle over S^1 (see [18]).

THEOREM 2. *For any* $n \geqq 9$ *there exists a neighborly combinatorial* 3-*dimensional Klein bottle with* n *vertices which is invariant under the vertex transitive action of the dihedral group* D_n. *Notation: type* II_n.

REMARK. (i) II_9 and II_{10} are exactly the types N^9_{51} and N^{10}_{3631} of [4] and [2] respectively.

(ii) II_n can be realized as a subcomplex of the cyclic polytope $C(n,6)$.

(iii) If the topology of the manifold is assumed to be fixed then for $n \leqq 19$ and $n \neq 10$ this type II_n is combinatorially unique according to the following theorem.

THEOREM 3. *For* $n \leqq 19$ *there are exactly* (*up to relabeling*) *the following neighborly combinatorial* 3-*manifolds invariant under the vertex transitive action of the dihedral group* D_n:

(i) *type* I_n *for* $5 \leqq n \leqq 19$ (*see Theorem* 1),

(ii) *type* II_n *for* $9 \leqq n \leqq 19$ (*see Theorem* 2),

(iii) *four exceptional types* $\widetilde{II}_{10}$, III_{15}, IV_{17}, IV_{19} *with* 10, 15, 17, 19 *vertices respectively.*

REMARK. (i) The proof consists in applying the algorithm described in section 3.

(ii) $\widetilde{\mathrm{II}}_{10}$ is exactly type N^{10}_{3629} of [2]. II_{10} and $\widetilde{\mathrm{II}}_{10}$ are homeomorphic but combinatorially different: according to [2] they are distinguished by the determinant of their edge-valence matrix. III_{15} is a highly symmetrical triangulation of the 3-dimensional torus which is closely related to the rhombidodecahedral tessellation of euclidean 3-space. It has been discussed in detail in the previous paper [14]. IV_{17} and IV_{19} are both nonorientable. They are homeomorphic to each other but they do not belong to an infinite series like II_n, i.e. there is no corresponding type IV_{21}, IV_{23}, etc.

However, there is an infinite series of manifolds-with-boundary related to IV_{17} and IV_{19}. For the notion of a *combinatorial* 3-*manifolds-with-boundary* it is required that the link of each vertex is either a triangulated 2-sphere or triangulated 2-disc. Examples of 3-manifolds-with-boundary are the 3-ball and the solid torus which is a product of a 2-disc with a circle (cf. [19]).

THEOREM 4. *For any* $n \geqq 7$ *there exists a neighborly combinatorial solid torus* *with n vertices which is invariant under the vertex transitive action of the dihedral group D_n. It may be chosen to be a subcomplex of the type I_n. Notation: type I'_n.*

THEOREM 5. (i) *For any odd $n \geqq 11$ there exists a nonorientable neighborly combinatorial 3-manifold-with-boundary which is invariant under the vertex transitive action of the dihedral group D_n. Its topology may be described to be the product of a Möbius band and a circle. It may be chosen to be a subcomplex of the type II_n. Notation: type II'_n.*

(ii) *For any odd $n \geqq 11$ there exists a neighborly combinatorial 3-manifold-with-boundary of the same topological type as II'_n which is invariant under the vertex transitive action of the dihedral group D_n. Notation: type IV'_n.*

REMARK. The types II'_{11} and IV'_{11} are combinatorially equivalent, II'_n and IV'_n are combinatorially different for any odd $n \geqq 13$. IV'_{17} and IV'_{19} happen to be subcomplexes of IV_{17} and IV_{19}. Therefore the types IV'_{11}, IV'_{17} and IV'_{19} can be closed up by adding one additional orbit leading to the manifolds (without boundary) II_{11}, IV_{17} and IV_{19}. This is impossible for the remaining types IV'_{13}, IV'_{15} and IV'_n, $n \geqq 21$.

THEOREM 6. *The numbers of combinatorially different neighborly combinatorial 3-manifolds with $n \leqq 15$ vertices which are invariant under the vertex transitive action of the cyclic group C_n are as follows:*

$n \leqq 8$:	1 *type*	(*with dihedral group D_n*),
$n = 9$:	2 *types*	(*both dihedral*),
$n = 10$:	5 *types*	(3 *dihedral*),
$n = 11$:	4 *types*	(2 *dihedral*),
$n = 12$:	10 *types*	(2 *dihedral*),
$n = 13$:	8 *types*	(2 *dihedral*),
$n = 14$:	22 *types*	(2 *dihedral*),
$n = 15$:	18 *types*	(3 *dihedral*).

A complete list of all these types will be given below in section 5.

2. Proof of Theorem 2: Spherical modifications of cyclic polytopes

By Gale's evenness condition the boundary complex of the cyclic polytope $C(n,4)$ is given by the following generating tetrahedra,

$$0123\ 0134\ 0145\ 0156\ \cdots\ 01\left[\frac{n}{2}\right]\left[\frac{n}{2}\right]+1,$$

where we denote the vertices by the integers modulo n. Each tetrahedron generates an *orbit* under the action of $T = (0\ 1\ 2\ 3\ \cdots\ n-1)$. Each such orbit is also invariant under the action of R. The word "orbit" here is motivated by the corresponding notion in the topological literature: group action on a topological space. In the combinatorial literature also the notion of "difference cycle" is used (cf. [10]).

The numbers f_i of i-dimensional simplices are as follows:

$$f_0 = n, \qquad f_1 = \binom{n}{2} = \frac{n}{2}(n-1), \quad f_2 = n(n-3), \quad f_3 = \frac{n}{2}(n-3).$$

It is well known that this complex is a neighborly combinatorial 3-sphere for any $n \geqq 5$. This leads to Theorem 1 which occurs in this paper only for completeness. We denote this combinatorial type by I_n because it is the first example which has been found (cf. [8] for the history of the cyclic polytopes). Figure 1 shows the

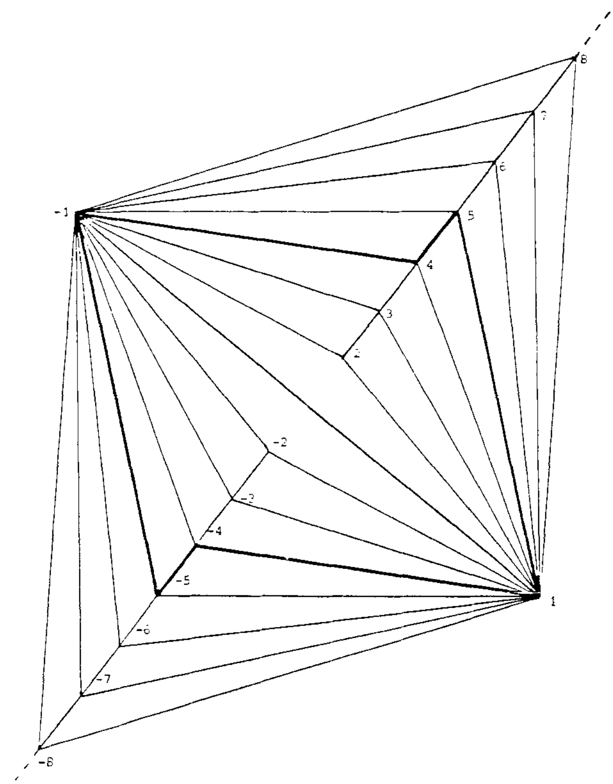


Fig. 1.

Schlegel diagram of the link of the vertex 0 in I_n (which is nothing but the boundary of the cyclic 3-polytope with $n-1$ vertices). The notation -1 instead of $n-1$, etc., is motivated by the symmetry R which here appears just as change of sign.

PROOF OF THEOREM 2. The series II_n for $n \geqq 9$ is simply given by a slight modification of I_n. This is called "spherical modification" or "surgery" in the topological literature (cutting out a manifold with boundary and replacing it by another one with the same boundary). We replace in I_n the three orbits of

$$0123 \ 0134 \ 0145$$

by the three orbits of

$$0135 \ 0245 \ 0235.$$

We observe that for $n \geqq 11$ both are 3-manifolds with the common boundary which is the torus given by the orbits of the triangles 015 and 045. Therefore II_n is given by the following generating tetrahedra:

$$0135 \ 0245 \ 0235 \ 0156 \ 0167 \cdots 01 \left[\frac{n}{2}\right]\left[\frac{n}{2}\right]+1.$$

This is in fact a neighborly combinatorial 3-manifold as follows from Fig. 2, which shows the Schlegel diagram of the link of the vertex 0 in II_n: observe the modification leading from Fig. 1 to Fig. 2.

So far the construction of II_n is complete. The required automorphism group is clear by construction. Furthermore by construction II_n it may be considered to be a subcomplex of the cyclic polytope $C(n,6)$ because the major part is already a part of $C(n,4)$ and the three orbits of 0135 0245 and 0235 are contained in the orbit of 012345 which is part of $C(n,6)$ by Gale's evenness condition. This is best possible because no nonorientable 3-manifold can be a subcomplex of the boundary of any 5-polytope. We show the nonorientability by contradiction. Suppose that II_n is orientable and assume that 0135 is an oriented tetrahedron. Then we get necessarily 2035 and 0245. The square of T^{-1} necessarily preserves the orientation (if it exists) and carries 0245 to the oriented tetrahedron $(-2)023$. Now the triangle 023 occurs twice with the same orientation, a contradiction.

It remains to determine the topological type of II_n. This has already been done in [11] for II_9, and in [2] it is mentioned that II_9 and II_{10} have the same topological type. For $n \geqq 11$ the spherical modification above leads to a decomposition of II_n into two manifolds with common boundary: the three orbits of 0135 0245 0235 and the orbits of

$$0156\ 0167 \cdots 01\left[\frac{n}{2}\right]\left[\frac{n}{2}\right]+1.$$

It is easy to see that the orbits of 0123 0134 0145 form a solid torus (i.e. a 3-manifold of the topological type of the product of a 2-disc with a circle) and that similarly its complement in I_n forms a solid torus (Hopf decomposition of the 3-sphere).

We illustrate here the method of *collapsing* which will be used throughout this paper (for basic facts compare [7]). Let us consider the three orbits of 0123 0134 0145. Here the orbit of 0145 collapses onto the two others (i.e. each single tetrahedron in the orbits does), then the orbit of 0134 collapses onto the orbit of 0123 which collapses on the orbit of the triangle 012 and finally on the orbit of the edge 01, the latter forming a polygonal circle. Therefore its homotopy type is that of a circle, its boundary is easily recognized to be a torus. Therefore the three orbits of 0123 0134 0145 form a solid torus for $n \geqq 11$. The same argument shows that the other orbits in I_n form a solid torus.

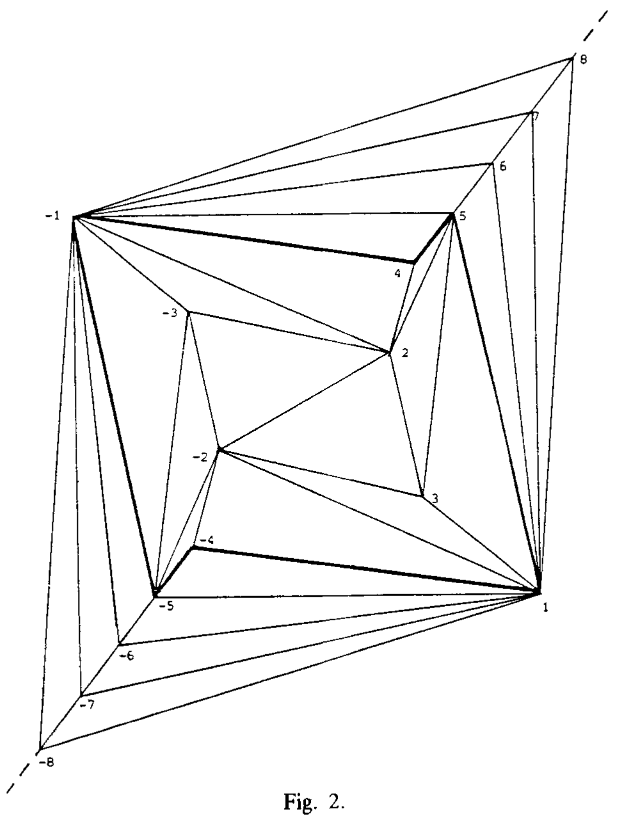


Fig. 2.

Now we are going to apply this method to the three orbits of 0135 0245 0235. As shown above this is for $n \geqq 11$ a nonorientable manifold whose boundary is the same torus given by the orbits of 015 and 045. Now the orbits of 0135 and 0245 together collapse onto the last orbit (see Fig. 3) which collapses onto the two orbits of 025 and 035 forming a torus, the "middle torus". The projection from the boundary torus onto this middle torus is a twofold covering. Figure 4 shows a "window" of this covering in two steps.

Therefore the manifold given by the tree orbits of 0135 0245 and 0235 is topologically the total space of a nontrivial line bundle over this middle torus. These line bundles are classified by the first homology group of the torus with integer coefficients mod 2. Therefore there are 4 different such bundles. By symmetry arguments there are only two different total spaces: The product of a cylinder and a circle for the trivial bundle, the product of a Möbius band and a circle for the three nontrivial bundles.

We have seen that II_n consists of two manifolds-with-boundary: a solid torus (call it M_1) and a product of a Möbius band with a circle (call it M_2). The fundamental group of M_1 is free with one generator a, the fundamental group of M_2 is free abelian with two generators b and c where b corresponds to the circle factor and c corresponds to the Möbius band factor. Then the fundamental group of the boundary torus will be free abelian with the two generators b and $2c$. It is not hard to see that a and $2c$ both can be represented by the same polygon in the boundary torus. The other generator b is homotopic to zero in M_1 and does therefore not occur in II_n: The fundamental group of II_n is free with

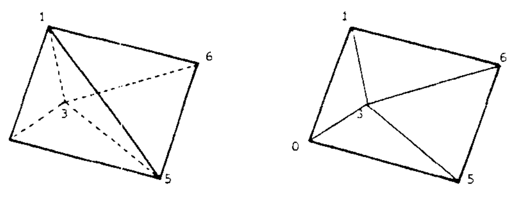
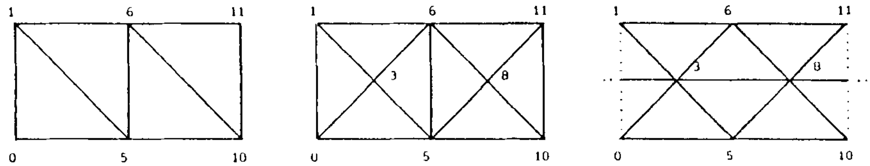


Fig. 3.



4 triangles of the boundary torus

after collapsing of 0135 and 1356

8 triangles of the middle torus

Fig. 4.

one generator. Therefore up to isotopy the identification of M_1 and M_2 along the common boundary is as in the following standard model for this situation: Consider the 3-dimensional Klein bottle $K^3 := S^2 \times [0,1]/\sim$ where $(x,0) \sim (rx,1)$, r being the reflection at the equatorial plane of S^2. In K^3 there sits the equatorial torus $S^1 \times [0,1]/\sim$ where r restricts to the identity. A tubular neighborhood of this torus in K^3 is the total space of a nontrivial line bundle over this torus. This corresponds to the part M_2. The generators b and c correspond to the first and the second factor of the equatorial torus respectively. The complement of this tubular neighborhood in K^3 is a solid torus and corresponds to M_1. This completes the proof that the underlying space of the simplicial complex II_n is homeomorphic to the 3-dimensional Klein bottle.

REMARK. In terms of piecewise linear Morse theory (cf. [5] [15]) the natural labeling of the vertices $0, 1, 2, \ldots, n-1$ induces a PL-Morse-function with only 4 critical points: minimum 0, maximum $n-1$ and two saddle points 3 and $n-3$. The middle level $(n-1)/2$ decomposes II_n into two solid Klein bottles, one below, one above this level.

3. Proof of Theorems 3 and 6: The enumeration algorithm

In this section we describe an algorithm for the enumeration of all neighborly combinatorial 3-manifolds with the prescribed kind of automorphism group (cyclic or dihedral). A similar algorithm has been used in a previous paper by the authors (see [13]): It puts together tetrahedra and checks certain necessary conditions. In the case to be discussed here the algorithm first enumerates all orbits of tetrahedra under the given group action and then puts together orbits of tetrahedra.

3.1. The Enumeration of all Orbits of Tetrahedra

We regard the vertices to be the elements of the group $\mathbf{Z}_n$ of integers modulo n. On this set there acts the cyclic group C_n by $T = (0\,1\,2\,3 \cdots n-1)$ and the dihedral group D_n by T and $R = (1\;n-1)(2\;n-2)(3\;n-3)\ldots$. Now all tetrahedra (= unordered 4-tuples of elements of $\mathbf{Z}_n$) are divided into equivalence classes (orbits) under the action of T or T and R respectively. Note that the action of T appears as the translation $x \curvearrowright x+1$ and the action of R as the reflection $x \curvearrowright -x$ in $\mathbf{Z}_n$. In the literature those orbits are also called "4-difference-cycles over $\mathbf{Z}_n$" (cf. [10]). Each orbit can be represented by the minimal tetrahedron in the lexicographical order, e.g. the orbit of 1245 in the case $n = 9$ is represented by 0134. Because the tetrahedra will be put together along triangles it is useful to calculate similarly the orbits of triangles (= unordered triples of elements of $\mathbf{Z}_n$) which appear as faces of an orbit of tetrahedra (they are also called "Heffter triples" in the literature). As an example the tetrahedron 0123 contains the triangles 012, 013, 023, 123 which belong to three different T-orbits 012, 013, 01 $(n-2)$.

In a combinatorial 3-manifold no triangle can be a face of three or more different tetrahedra. Therefore if it happens that the orbit of a triangle occurs three or more times in the orbit of a tetrahedron then we have to erase this orbit from the list of all orbits of tetrahedra.

3.2. The Building Algorithm

This algorithm uses two different kinds of higher data structures:

1. The *list* of all equivalence classes of 4-tuples as described in 3.1 above.

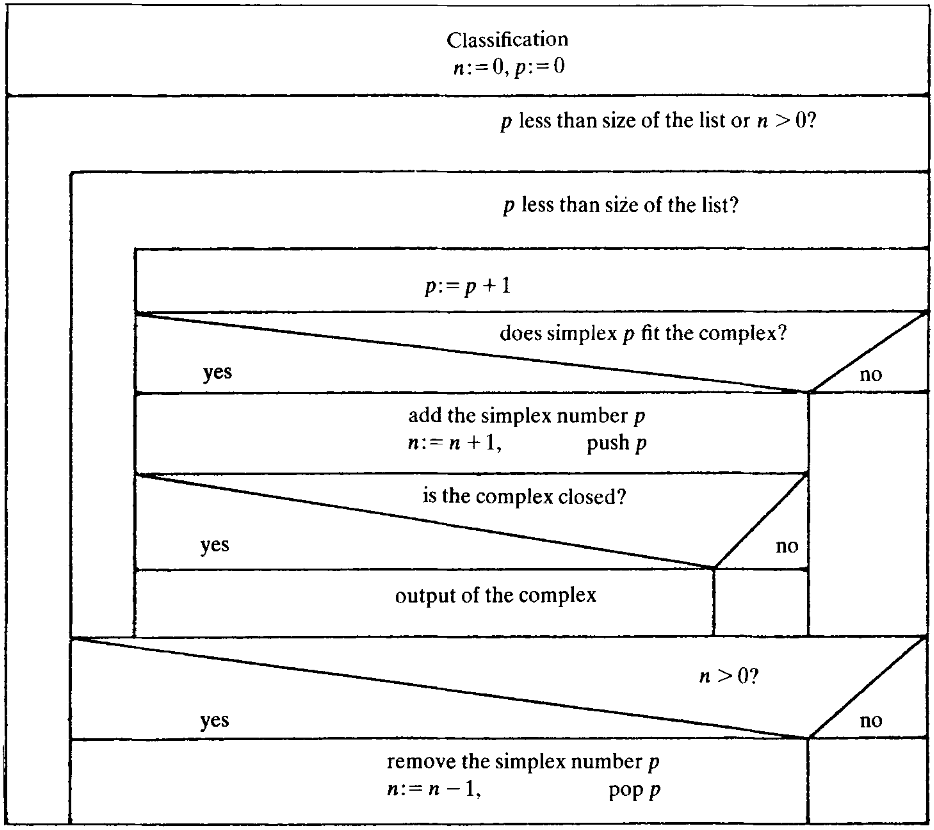
2. A so-called *complex* which consists of a list of equivalence classes (orbits) of 4-tuples together with a list of the equivalence classes (Heffter triples) of triangles which occur in the complex once ("free" triangle) or twice ("closed" triangle).

A complex is called *closed* if all of its triangles are closed in the sense that they occur twice each. The following important *parameters* are used: n is the actual number of classes in the complex, p is the actual pointer on the list of classes pointing to the most recently checked class. Both parameters are zero at the beginning. The main part of the program is represented by the so-called *Nassi–Shneiderman diagram* (Table 1). From this description it should be clear that the algorithm will find all possible closed 3-complexes which by construction are invariant under the prescribed group action. Furthermore it has to be checked if such a complex contains all possible edges (neighborliness) which is a purely combinatorial condition. Finally one has to check if it is a combinatorial manifold, i.e. if the vertex link is a combinatorial 2-sphere or not. This completes the enumeration of all such combinatorial 3-manifolds.

It remains to check which of them are combinatorially different. Basically every permutation of the n vertices applied to a complex leads to an isomorphic one. Some of those permutations will even be compatible with the group action, i.e. they will preserve the orbits. The following lemma says that these are exactly the affine transformations of $\mathbf{Z}_n$.

LEMMA. *Let P be a permutation of the n elements of $\mathbf{Z}_n$. Assume that for each $m \in \mathbf{Z}_n$, PT^mP^{-1} is some power of T. Then P is an affine transformation of the form $P(x) = ax + b$, $a, b, x \in \mathbf{Z}_n$, a being a unit, where $b = P(0)$, $a = PTP^{-1}(0)$.*

TABLE 1.
The Nassi–Shneiderman Diagram



PROOF. First of all if $PTP^{-1} = T^k$ for some k then necessarily $PTP^{-1}(0) = k$. Consequently $PT^mP^{-1} = (PTP^{-1})^m = T^{km}$ and for every $x \in \mathbf{Z}_n$, $P(x) = P(T^x(0)) = T^{kx}(P(0)) = kx + P(0)$.

REMARKS. (i) k must be a unit in the ring $\mathbf{Z}_n$ because otherwise P cannot be a permutation.

(ii) The same holds if we formulate the assumption for elements of the dihedral group instead of the cyclic group consisting of all powers of T. In this case the reflection R appears as the transformation $x \mapsto -x$.

(iii) Usually the multiplicative group of all units in the ring $\mathbf{Z}_n$ is denoted by $\mathbf{Z}_n^*$. There is a natural action of $\mathbf{Z}_n^*$ on $\mathbf{Z}_n$ and on all orbits of tuples: $(a, [x]) \mapsto [ax]$. If one likes there is an exact sequence

$$0 \to \mathbf{Z}_n \to A(1, \mathbf{Z}_n) \to \mathbf{Z}_n^* \to 1$$

where $A(1,\mathbf{Z}_n)$ denotes the affine group in one dimension over $\mathbf{Z}_n$. It is well known from elementary algebraic number theory that the order of $\mathbf{Z}_n^*$ is just the Eulerian ϕ-function

$$\phi(n) = \sum_{\substack{m \leqq n \\ (m,n)=1}} 1 = n \cdot \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

Consequently the algorithm described above will find for each such combinatorial 3-manifold also all 3-manifolds equivalent under this action of $\mathbf{Z}_n^*$. It might happen that some manifolds are invariant under some elements of $\mathbf{Z}_n^*$. For example, the type III_{15} is invariant under the action of the full affine group $A(1, Z_{15})$ (see [14]).

3.3. The Result

The combinatorially different neighborly 3-manifolds with at most 19 vertices admitting a vertex transitive action of the dihedral group have been found to be the following (each type is given by the list of generators of its T-orbits):

$n \geqq 5$:	type I_n	0123 0134 0145 0156 0167 ... 01 $\left[\frac{n}{2}\right]\left[\frac{n}{2}\right]+1$
$n \geqq 9$:	type II_n	0135 0235 0245 0156 0167 ... 01 $\left[\frac{n}{2}\right]\left[\frac{n}{2}\right]+1$
$n = 10$:	type $\widetilde{\mathrm{II}}_{10}$	0136 0138 0156 0158
$n = 15$:	type III_{15}	0137 013 11 0157 015 13 019 11 019 13
$n = 17$:	type IV_{17}	0134 0145 0156 013 10 016 12 018 10 018 15
$n = 19$:	type IV_{19}	0134 0145 0156 0167 013 11 017 13 019 11 019 17.

REMARK. Each tetrahedron contains two subsequent vertices $i\ i+1$. According to Gale's evenness condition each of the combinatorial manifolds above may be regarded as a subcomplex of the cyclic polytope $C(n,6)$. For the nonorientable types II_n, $\widetilde{\mathrm{II}}_{10}$, IV_{17}, IV_{19} this embeddability into a simplicial 5-sphere (or euclidean 5-space) is best possible: no embedding can exist into the 4-sphere (or 4-space). For the type III_{15} we don't know if this can be realized in the boundary complex of any 5-polytope.

4. Proof of Theorems 4 and 5: The case of manifolds-with-boundary

To obtain the type I'_n one simply has to remove the orbit of 0123 from I_n. The boundary of this orbit (and therefore of its complement in I_n) is the torus given by the orbits of the triangles 013 and 023. These tori have been observed already by A. Altshuler (see [1]). As in section 2 above it is easily seen that the orbit of 0123 forms a solid torus, just as its complement I_n'. I_n' is also neighborly because no edges are lost by removing the orbit of 0123. This proves Theorem 4.

Note that for odd $n = 2k + 1$ one also could remove the orbit of $01k\ k+1$ instead of 0123. The resulting combinatorial manifolds-with-boundary are combinatorially equivalent.

Similarly for odd $n = 2k + 1$ we define II_n' by removing the orbit of $01k\ k+1$ from II_n. Topologically we remove a solid torus. It is easy to see that II_n' collapses onto the three orbits of 0135 0245 0235 where the topology of the boundary torus remains unchanged. This implies that II_n' is homeomorphic to the collection of those three orbits. In the proof of Theorem 2 above we have already seen that the latter is homeomorphic to the product of a Möbius band and a circle. This proves part (i) of Theorem 5.

The construction of IV_n' for $n = 2k + 1$ is motivated by IV_{17} and IV_{19} (see section 3.3). We define IV_n' to be the collection of the orbits of 0134 0145 0156 ... $01\ k-3\ k-2\ 013-(k-1)\ 01\ k(-2)\ 01\ k-(k-1)$ where we wrote -2 instead of $n-2$, etc. The link of the vertex 0 is the triangulated disc shown in Fig. 5. Therefore IV_n' is in fact a combinatorial 3-manifold-with-boundary.

From Fig. 5 it can be seen that IV_n' is constructed by a "spherical modification" from I_{n-4}' similarly as II_n is constructed from I_n. Therefore the topology of IV_n' can be understood following the pattern of the proof of Theorem 2. All that has to be shown is that the collection of the three orbits of $01\ k-(k-1)$ $013-(k-1)$ and $01k\ -2$ is topologically the product of a Möbius band with a circle. Just as was used in the proof of Theorem 2 we have the collapsing of the

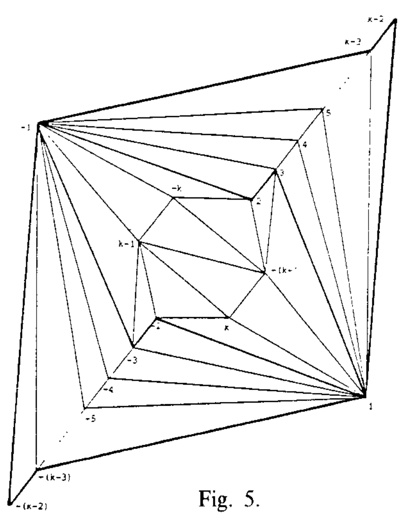


Fig. 5.

orbits of $01k-2$ and $013-(k-1)$ together onto the orbit of $01k-(k-1)$ which collapses onto the torus formed by the orbits of $0k-(k-1)$ and $0k-2$, the middle torus. The projection from the boundary torus onto this middle torus is a double covering just as shown in Fig. 4. We only have to change the labeling to get Fig. 6, showing a "window" of this covering.

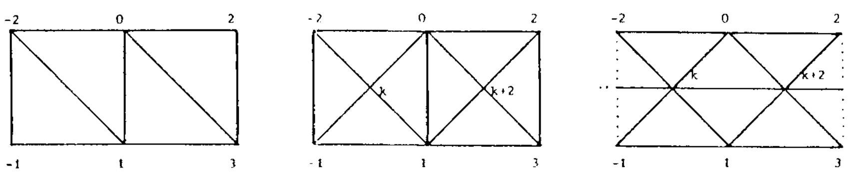
Now as in the proof of Theorem 2 it follows that IV'_n is homeomorphic to the product of a Möbius band and a circle. This proves part (ii) of Theorem 5.

Now we are going to prove the remarks after Theorem 5.

The two types II'_{11} given by the orbits of 0135 0245 0235 and IV'_{11} given by 0137 0159 0157 are combinatorially equivalent by applying to the first one the permutation $x \curvearrowright 2x$ modulo 11.

For odd $n \geqq 13$, II'_n and IV'_n are combinatorially inequivalent because already their links of vertices are different: The vertex link in II'_n contains four vertices of valence 6 (i.e. 6 edges meeting) whereas the vertex link in IV'_n contains only two vertices of valence 6 (see Figs. 2 and 5).

The type IV'_{2k+1} consists of $k-2$ orbits, i.e. $(2k+1)(k-2)$ tetrahedra. A neighborly combinatorial 3-manifold without boundary having $n=2k+1$ vertices has necessarily $n(n-3)/2=(2k+1)(k-1)$ tetrahedra. This means that there is only one chance to close up IV'_n to get a neighborly combinatorial manifold IV_n without boundary and with the same kind of symmetry: try to add exactly one T-orbit which in addition has to be invariant under R. Now the boundary of IV'_n in the link of 0 appears as the hexagon shown in Fig. 7.



4 triangles of the boundary torus

after collapsing of $01k(-2)$ and $013-(k-1)$

8 triangles of the middle torus

Fig. 6.

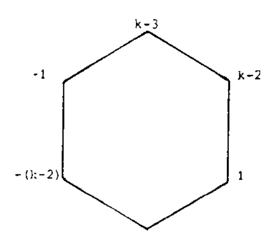


Fig. 7.

Each T-orbit of tetrahedra gives four triangles in the link of 0. Let us look for all possibilities to fill this hexagon by four triangles where $k \geqq 5$ is an arbitrary integer:

1st case: Introduce the tetrahedron $0\,1\,k-3\,k-2$. By T-action we get

$$\begin{array}{llll} 0 & 1 & k-3 & k-2 \\ -1 & 0 & k-4 & k-3 \\ -(k-3) & -(k-4) & 0 & 1 \\ -(k-2) & -(k-3) & -1 & 0 \end{array}$$

Therefore $\pm(k-4)$ both must be among the vertices ± 1, $\pm(k-2)$, $\pm(k-3)$ which is possible only for $k=5$.

2nd case:

$$\begin{array}{llll} 0 & k-3 & k-2 & -1 \\ -(k-3) & 0 & 1 & -(k-2) \\ -(k-2) & -1 & 0 & -(k-1) \\ 1 & k-2 & k-1 & 0 \end{array}$$

This implies that $\pm(k-1)$ both must be among ± 1, $\pm(k-2)$, $\pm(k-3)$ which is impossible.

3rd case:

$$\begin{array}{llll} 0 & k-3 & -(k-2) & -1 \\ -(k-3) & 0 & 6 & -(k-2) \\ k-2 & -6 & 0 & k-3 \\ 1 & k-2 & -(k-3) & 0 \end{array}$$

This implies that ± 6 both must be among ± 1, $\pm(k-2)$, $\pm(k-3)$ which is possible exactly for $k=8$ and $k=9$.

It follows that IV'_n can be closed up by one additional orbit at most for $n=11$, $n=17$ and $n=19$. Vice versa, in these three cases it is possible: IV'_{11}, IV'_{17} and IV'_{19} are subcomplexes of II_{11}, IV_{17} and IV_{19} respectively.

Finally we discuss roughly the topology of IV_{17} and IV_{19}. We have seen above that IV'_{17} and IV'_{19} both are homeomorphic to the types II'_n. To close IV'_{17} and IV'_{19} we put in the orbit of the tetrahedra 0156 or 0167 respectively, each forming a solid torus. This means that IV_{17} and IV_{19} both are spherical modifications of the 3-dimensional Klein bottle where each time the same kind of modification is used. In terms of the generators a, b, c in section 2 it seems that here a is homotopic to b and $2c$ is homotopic to zero. This implies that the fundamental group is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_2$. The PL-function induced by the standard ordering $0, 1, 2, \ldots, 16$ or $0, 1, 2, \ldots, 18$ has in fact only 6 critical points.

5. Appendix: Neighborly combinatorial 3-manifolds with cyclic automorphism group

In this last section we want to discuss the analogous problem when the dihedral group D_n is replaced by the cyclic group C_n, i.e. we assume that the cyclic group generated by T acts transitively on the n vertices. In this case the same algorithm can be applied (cf. section 3). The number of such combinatorial manifolds is much larger than that in the case of the dihedral group. This makes it difficult to classify the topological types of all of them. Before giving the list of the combinatorial types let us mention at least one infinite series of such manifolds (similarly to Theorem 2):

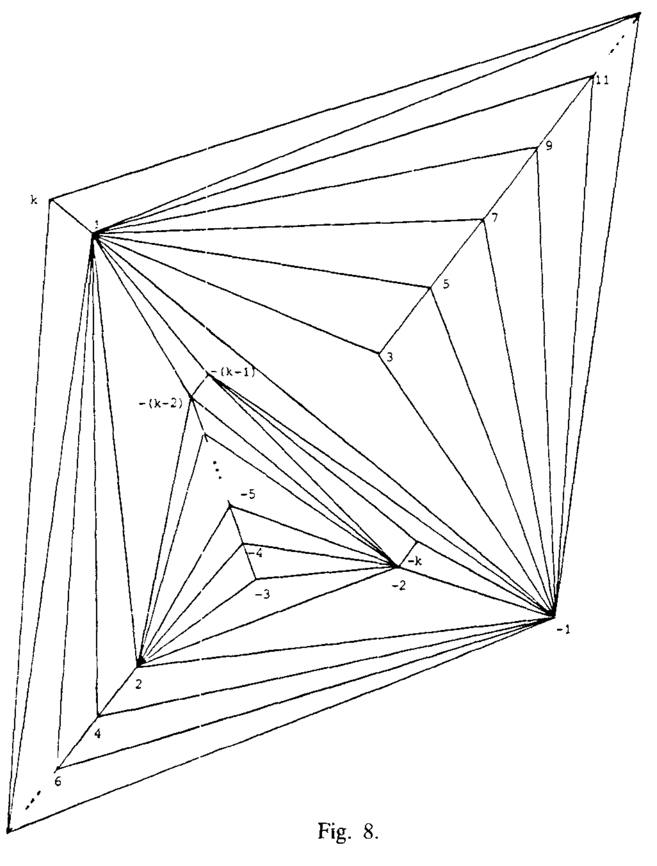


Fig. 8.

TABLE 2.
Complete List of Combinatorial Types of Neighborly 3-Manifolds Admitting a Vertex Transitive Action of the Cyclic Group C_n

n	Type	Orientation	List of Orbits						Remarks
$\leqq 8$	I_n	yes							
9	I_9	yes	0123	0134	0145				
	II_9	no	0135	0235	0245				N^9_{51} in [4]
10	I_{10}	yes	0123	0134	0145	0156			
	II_{10}	no	0135	0235	0245	0156			N^{10}_{3631} in [2]
	$\widetilde{\mathrm{II}}_{10}$	no	0136	0138	0156	0158			N^{10}_{3629} in [2]
	1_{10}	yes	0124	0125	0135	0257			N^{10}_{3574} in [2]
	2_{10}	yes	0125	0126	0146	0257			N^{10}_{3611} in [2]
11	I_{11}	yes	0123	0134	0145	0156			
	II_{11}	no	0135	0235	0245	0156			
	1_{11}	yes	0123	0137	0147	0149			
	2_{11}	no	0124	0128	0135	0145			cf. Theorem 7
12	I_{12}	yes	0123	0134	0145	0156	0167		
	II_{12}	no	0135	0235	0245	0156	0167		
	1_{12}	yes	0123	0137	0157	0268	01510		
	2_{12}	yes	0125	0128	0145	0179	0268		
	3_{12}	no	0134	0139	0146	0167	01710		
	4_{12}	no	0134	0139	0147	0167	01610		
	5_{12}	yes	0136	0137	0167	0246	0258		
	6_{12}	yes	0136	01310	0146	01410	0268		
	7_{12}	yes	0137	0139	0147	0149	0268		
	8_{12}	yes	0137	0139	0167	0169	0246		
13	I_{13}	yes	0123	0134	0145	0156	0167		
	II_{13}	no	0135	0235	0245	0156	0167		
	1_{13}	no	0123	0137	0157	01511	0268		
	2_{13}	no	0123	0137	0159	01511	0179		
	3_{13}	no	0123	0138	0156	01511	0169		
	4_{13}	yes	0124	0125	0138	0158	0247		
	5_{13}	no	0124	0129	0135	0146	0156		cf. Theorem 7
	6_{13}	no	0125	0126	0146	0157	0268		
14	I_{14}	yes	0123	0134	0145	0156	0167	0178	
	II_{14}	no	0135	0235	0245	0156	0167	0178	

THEOREM 7. *For each odd $n \geqq 11$ there exists a neighborly combinatorial 3-dimensional Klein bottle with n vertices admitting a vertex transitive action of the cyclic group C_n (but not of D_n).*

SKETCH OF PROOF. Let $n = 2k + 1$. This manifold is defined to be the collection of the T-orbits of the following tetrahedra:

TABLE 2 (contd.)

n	Type	Orientation	List of Orbits						Remarks
	1_{14}	yes	0123	0137	0157	015 12	0279	0246	
	2_{14}	yes	0123	0137	0157	015 12	0279	0268	
	3_{14}	yes	0123	0137	0157	015 12	0279	026 10	
	4_{14}	yes	0123	0137	017 12	0279	027 10	0246	
	5_{14}	yes	0123	0137	017 12	0279	027 10	0268	
	6_{14}	yes	0123	0137	017 12	0279	027 10	026 10	
	7_{14}	yes	0123	0137	015 10	015 12	0178	018 10	
	8_{14}	yes	0123	0138	0158	015 12	0369	037 10	
	9_{14}	yes	0124	0127	0135	0146	0157	0279	
	10_{14}	yes	0125	0126	0136	0137	0147	037 10	
	11_{14}	yes	0125	0126	0146	0257	0279	0246	
	12_{14}	yes	0125	0126	0146	0257	0279	0268	
	13_{14}	yes	0125	0126	0146	0257	0279	026 10	
	14_{14}	no	0125	0129	0145	018 10	019 11	0279	
	15_{14}	no	0125	012 10	0146	0156	0257	0279	
	16_{14}	yes	0126	0127	0135	0137	024 11	0279	
	17_{14}	yes	0126	0127	0147	014 12	015 12	037 10	
	18_{14}	yes	0126	0127	015 11	017 11	0269	0279	
	19_{14}	no	0134	0137	0145	015 12	0178	018 10	
	20_{14}	no	0134	0138	0145	015 12	0178	017 10	
15	I_{15}	yes	0123	0134	0145	0156	0167	0178	
	II_{15}	no	0135	0235	0245	0156	0167	0178	
	III_{15}	yes	0137	013 11	0157	015 13	019 11	019 13	cf. [14]
	1_{15}	no	0123	0134	0149	0167	016 12	017 10	
	2_{15}	yes	0123	0136	016 13	0258	037 10	037 11	
	3_{15}	yes	0123	0137	014 11	014 13	017 11	036 10	
	4_{15}	no	0123	0137	0157	015 13	0268	0279	
	5_{15}	no	0123	0137	017 13	0259	025 11	026 11	
	6_{15}	no	0123	0139	0149	014 10	016 10	016 13	
	7_{15}	yes	0123	0139	0159	015 13	027 11	028 11	
	8_{15}	yes	0123	0139	0169	016 13	037 10	037 11	
	9_{15}	yes	0123	013 11	0147	014 13	017 11	026 10	
	10_{15}	no	0123	013 13	0259	0268	026 12	028 11	
	11_{15}	no	0124	012 10	0135	0146	0157	0167	cf. Theorem 7
	12_{15}	no	0124	012 10	013 10	0149	0249	027 10	
	13_{15}	no	0125	0126	0146	0257	0268	0279	
	14_{15}	no	0126	0127	0137	0138	0158	037 10	
	15_{15}	no	0126	0127	015 13	017 13	0268	028 11	

$$012\,k+3 \qquad 0124\ 0135\ 0146\ 0157\ \ldots\ 01\,k-2\,k \qquad 01\,k-1\,k.$$

For $k = 4$ we get 0127 0124 0134 which is combinatorially equivalent to the type II_9 above. For $k = 5$ we have 0128 0124 0135 0145 which is no longer invariant under the action of R. By construction for each $k \geqq 5$ the resulting simplicial complex is invariant under the action of T (but not of R). That it is in fact a combinatorial 3-manifold follows from Fig. 8, which shows the link of the vertex 0. It also follows that it is nonorientable: look at the triangles $12\,k+3$ and $1\,-1\,k+2$ in the link of 0, both belonging to the T-orbit of $012\,k+3$. Therefore the T-action does not preserve the orientation of the tetrahedra. Because the number of vertices is odd the T-action must preserve a global orientation if it exists, a contradiction.

Along the pattern of the proof of Theorem 2 it can be seen that this type is also topologically a 3-dimensional Klein bottle. Again the PL-function induced by the standard labeling $0, 1, 2, \ldots, n-1$ will have only four critical points.

Now the complete list of the combinatorial types of neighborly 3-manifolds admitting a vertex transitive action of the cyclic group C_n is given in Table 2 (each type is given by the list of generators of the T-orbits).

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